J. Fluid Mech. (1966), *vol.* 25, *part* 3, *pp.* 457–467
Printed in Great Britain

On the theory of asymmetric shear flows past flat plates

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(Received 6 September 1965)

When a semi-infinite flat plate is immersed parallel to an unbounded, plane, steady, asymmetric, constant shear flow of an incompressible viscous fluid, an interaction occurs between the surface-generated vorticity and the external vorticity. A physical assumption was made in a previous paper (Mark 1962) concerning this problem that the pressure field in a thin layer adjacent to the top-side of the plate may be accurately approximated by the undisturbed constant-pressure field—that which exists before the insertion of the plate into the flow. This means that the vorticity interaction is assumed to have no effect on the undisturbed pressure field. That this is a valid first approximation far downstream along the plate where the interaction is intense is given rigorous support in this paper.

The flow below the plate is examined on a heuristic basis. It is found that there is a strong possibility for the flow to separate from the lower surface near the leading edge of the plate. However, far downstream the flow settles down to a Stokes-type flow near the plate.

1. Introduction

In this paper we shall add support to and extend the theory advanced by Mark (1962) (hereafter referred to as M) concerning the problem of a semi-infinite flat plate of zero thickness that is fixed parallel to an otherwise undisturbed, plane, unbounded, steady, asymmetric, constant shear flow of an incompressible viscous fluid. This is admittedly a highly idealized flow model of a body immersed in a stream with transverse total pressure gradients (or transverse entropy gradients), but it is nevertheless amenable to analytical treatment—the results of which may provide some preliminary insight, if not a fundamental understanding, of the actual flow phenomena. Problems of this kind have recently gained importance, particularly in connexion with the motion of blunt bodies at hypersonic speeds, in which strong entropy gradients exist across the shock layer, and it is thus useful to impart confidence to the methods that have been proposed for the treatment of such problems.

It is expedient to review in perspective the theory presented in M. First we pose the general flow problem mathematically as follows. The solution for the complete flow field around the plate is required to satisfy the full Navier–Stokes equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{1.1a}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} \right), \tag{1.1b}$$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} \right), \tag{1.1c}$$

and the boundary conditions:

$$u = v = 0 \quad \text{on the plate} \quad y = 0 \quad (x \geqslant 0), \tag{1.1d}$$

$$u \to U + \omega y, \quad v \to 0, \quad p \to P \quad \text{as} \quad x \to -\infty, \tag{1.1e}$$

where (u, v) are the velocity components in the (x, y)-directions of the usual rectangular co-ordinate system, p is the pressure, ρ the density, μ the viscosity coefficient, U the constant velocity component of the undisturbed flow, P the constant undisturbed pressure, and ω the constant positive external vorticity (the local vorticity in the flow is here defined as $\Omega = (\partial u/\partial y) - (\partial v/\partial x)$). It is important to notice that the asymmetry feature of the oncoming flow is due to the requirement, since a *viscous* fluid is considered, that Ω be mathematically continuous everywhere far upstream of the plate. Since, otherwise, the admission of a discontinuity in the vorticity is incompatible with the motion of a viscous fluid, because diffusion would immediately smooth out the discontinuity. (Even if the effects of viscosity are ignored in a theoretical model, there would be an infinity of possible solutions depending upon the value of Ω that may be arbitrarily prescribed along the surface of discontinuity.)

No such complete solution of the above system has been found, but limiting solutions may be obtained that are valid in certain regions of the flow. It was shown in M that, for values of x such that the local Reynolds number $Re_x = Ux/\nu$ (ν is the kinematic viscosity) is large, a limiting flow solution exists in a thin layer adjacent to the plate for $y \geqslant 0$. Qualitatively, some of the physical characteristics of the flow in such a thin layer are similar to those in the 'boundary layer' of the classical Prandtl–Blasius case ($\omega = 0$). For example, diffusion of surface-generated vorticity occurs predominantly within the thin layer, and this primarily in the direction normal to the plate; i.e. the transition from the surface-generated vorticity to the external vorticity ω is practically complete across the thin layer. As is well known, such behaviour is much like the diffusion of heat from a hot plate immersed parallel to a fast-moving stream (this stream must now be regarded as heated by an external energy source in order to preserve the analogy between external heat and external vorticity); i.e. the surface-generated vorticity, as it diffuses outwards from the plate, is immediately swept downstream along streamlines that do not penetrate appreciably into the mainstream but instead remain nearly parallel to the plate.

There exist important differences, however; so significant that we shall henceforth call the thin layer the *vorticity interaction layer* and denote it by D_v for brevity. For example, when the oncoming stream is initially endowed with vorticity, the tangential velocity profiles at two given stations in D_v are no longer similar as in the classical case $\omega = 0$. Also, the interaction between surface-generated vorticity and external vorticity causes an unexpected 'defect' Δu_δ ($= U + \omega y - u$) in the tangential velocity relative to the undisturbed velocity at the outer edge—which we denote by D_v^e—of D_v, figure 1. (As in classical boundary-layer theory, we must accept the fact that the outer edge of D_v cannot be located precisely, but, for practical purposes, it is convenient to define D_v^e as where transverse diffusion of surface-generated vorticity is negligible to within a prespecified amount.) Thus, if y is of the order of the thickness δ of D_v and δ^* represents the outward shift of the streamlines at D_v^e, then a fluid particle at a distance $(y-\delta^*)$ from the x-axis before the insertion of the plate, retains its x-momentum

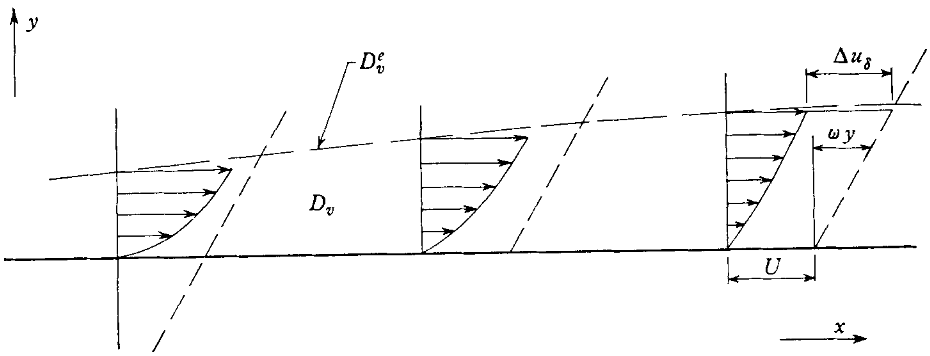

FIGURE 1. Velocity profiles across the vorticity interaction layer on the top-side of the plate.

when displaced by an amount δ^* to the position y after the insertion of the plate.† At D_v^e, $\Delta u_\delta = \omega\delta^*$, where δ^* is in general a function of the transverse 'viscous length' $\surd(\nu x/U)$ and the vorticity number $\xi = (1/L)\surd(\nu x/U)$, where $L = U/\omega$ is a transverse 'vortical length' (which may be interpreted as the transverse length across which the undisturbed tangential velocity changes by an amount U). When $\xi \to 0$, $\delta^* \to 1{\cdot}721\surd(\nu x/U)$, which is the classical Blasius expression in the limit; when $\xi \to \infty$, $\delta^* \to L$, which is a constant in the limit. Thus, we have approximately, $\Delta u_\delta = 1{\cdot}721 U\xi$ for $\xi \ll 1$ and $\Delta u_\delta = U$ for $\xi \gg 1$.‡

It should be mentioned that, in so far as the calculation of the flow external to D_v is concerned, the existence of a velocity defect at D_v^e means that we cannot simply regard D_v as due only to an effective source flow emanating from the plate with a vertical velocity and thus treat the external flow as inviscid. Rather, with respect to this external flow problem, an *additional* boundary condition on the tangential velocity must be satisfied in some common region between D_v and the unbounded domain external to it. This means, therefore, that it is necessary to retain the viscous terms in the equations governing the flow external to D_v—i.e. a *viscous* theory is necessary in this region. (Of course, if this point of view is adopted, it must be shown that the viscous terms external to D_v are of a lower order of importance than those within it.)

The whole of the theory presented in M for the flow in $y \geq 0$ is necessarily

† This physical phenomenon has a broader connotation than was earlier suspected; for example, it also occurs in the stagnation region of blunt bodies as observed in the exact solutions of Stuart (1959) in the plane case and Kemp (1959) in the axisymmetric case.

‡ The approximate result that $\Delta u_\delta = 1{\cdot}721 U\xi$ for $\xi \ll 1$ may be extracted from the earlier work of Goldstein (1930) in his section (3.12). Glauert (1957) later deduced it independently.

approximate when Re_x is finite, but, as in all asymptotic approximations, the error is expected to diminish uniformly for all points within D_v as Re_x increases. The theory does not unravel solely from this hypothesis, however, for in the course of the analysis it was necessary to introduce the simplifying assumption that the undisturbed pressure field is insensitive to the presence of D_v, at least to the first approximation. That is, the pressure was assumed to be *independent* of the effects of viscosity (hence of ξ) and equal to P *uniformly* in D_v to the first approximation. This assumption appears to be well justified on purely physical grounds, since it was demonstrated *a posteriori* in M that D_v steadily shrinks in thickness as both Re_x and ξ steadily increase, which in the limit signifies that the retardation of the flow within D_v cannot significantly disrupt the nearly tangential course of the streamlines external to D_v (i.e. they do not penetrate sharply into the external flow), and hence cannot cause a sensible pressure disturbance in a flow that is devoid of extraneous lateral constraints. In § 2 we shall supply a more rigorous justification of this assumption in the case of large ξ (we do not subscribe to complete mathematical rigour, however). It will be helpful physically to interpret increasing ξ as corresponding both to increasing 'intensity' of the vorticity interaction and to increasing distance downstream from the leading edge. Thus we will refer to the *weak vorticity interaction* case as occurring near the leading edge and the *strong vorticity interaction* case as occurring far downstream from the leading edge.

Due to the asymmetry of the oncoming flow, the flow below the plate ($y \leqslant 0$) is not so readily discernible intuitively as it was for the flow above the plate. In § 3 a heuristic approach will be adopted for the theoretical treatment of this bottom region, and it will be shown that an interesting flow phenomenon is possible.

2. Theoretical justification of the constant pressure assumption for strong vorticity interaction

We begin by elucidating what is meant by the *first approximation* to the flow within D_v on the top side of the plate. To do this in a systematic manner, we shall construct a special limit process for formally obtaining it. By so doing, we will have taken a necessary first step towards placing the first approximation within a general procedure for obtaining the exact solution of the problem, at least on a formal basis.

According to the results of M, the appropriate set of dimensionless dependent quantities for D_v is

$$\bar{u} = \frac{u-\omega y}{U}, \quad \bar{v} = \frac{v}{U}\sqrt{Re_x}, \quad \bar{p} = \frac{p-P}{\rho U^2}, \quad \bar{\Psi} = \frac{\Psi - \frac{1}{2}\omega y^2}{\sqrt{(U\nu x)}}$$

(Ψ is the dimensional stream function defined in the usual way), and the appropriate set of dimensionless independent quantities is

$$\xi^{-1} = L\sqrt{\frac{U}{\nu x}}, \quad \bar{y} = y\sqrt{\frac{U}{\nu x}}, \quad Re_x = \frac{Ux}{\nu}.$$

When the physical quantities used in the description of the flow within D_v have been scaled in this manner, we shall refer to the transformed system as being placed on the D_v-scale.

Next we define the appropriate limit process for D_v as

$$\lim_v\colon Re_x \to \infty \quad \text{with } (\xi, \bar{y}) \text{ fixed in } (0 < \xi < \infty, 0 \leqslant \bar{y} < \infty).$$

Finally we define the following sequence of operations—which we denote by S_v:

. Place the physical quantity or relation on the D_v-scale.

.. Apply $\lim_v$ to the transformed quantity or relation.

... Transform the limit quantity or relation back to the physical system.

Therefore, by applying S_v to (1.1 *a, b, c*) we obtain in a straightforward manner the first approximation for the equations governing the flow within D_v as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{2.1b}$$

$$\partial p/\partial y = 0. \tag{2.1c}$$

Although this limiting system is outwardly of the same form as the classical boundary-layer system, it nevertheless differs fundamentally from the latter in that the vorticity interaction phenomenon is preserved through the fixing of the vorticity number ξ under $\lim_v$.† To understand the import of this better, it is seen that both transverse length dimensions—y and L—have been scaled by the same factor (inverse of the viscous length), but their ratio y/L remains unchanged. In particular, suppose that the co-ordinate y corresponds to D_v^e, i.e. suppose $y = \delta$. Then using the results from M we have $\delta/L \sim \xi$ when $\xi \ll 1$ and $\delta/L \sim \xi^{\frac{2}{3}}$ when $\xi \gg 1$, so that fixing ξ under $\lim_v$ means fixing y/L when $y = \delta$. In other words, the *ratio* of the thickness of D_v to the vortical length is required to be *invariant* both to the scaling operation and to an increase in the Reynolds number Re_x.

As in the classical case, (2.1*c*) states the remarkable fact that, at a given station along the plate, the pressure in D_v is independent of the distance y from the plate, despite the varying retardation of the flow across D_v due to the action of friction. A direct consequence of this is that the pressure at the wall is now equivalent to the pressure at D_v^e.

In order to further clarify the role of the constant pressure assumption, we present a different formulation of the required *asymptotic* boundary condition at D_v^e than that given in M. In the first place, such an asymptotic condition must be consistent with, and hence derivable from, the system (2.1), since the fluid is actually in motion at D_v^e and is not constrained there in any manner. (If extraneous lateral constraints are present, it is likely that a pressure gradient would be induced along the plate.) Now it may be justified *a posteriori* that the surface-generated vorticity decays exponentially as D_v^e is approached, i.e. transverse diffusion of surface-generated vorticity is negligible there—being occupied primarily by external vorticity. Thus, with the auxiliary asymptotic conditions

† In conjunction with a suggestion made in § 1, the introduction of the novel limit process $\lim_v$ implies that a different *external* limit process other than the usual one must be constructed if successive approximations to the flow external to D_v are to be compatible with those within D_v.

$\partial^2 u/\partial y^2 \to 0$ and $\partial u/\partial y \to \omega$ as D_v^e is approached (i.e. as $\bar{y} \to \infty$) and setting $v = -\partial\Psi/\partial x$, (2.1*b*) yields

$$\frac{\partial \Phi}{\partial x} \to 0 \quad \text{as} \quad \bar{y} \to \infty,$$

where $\Phi = \frac{1}{2}u^2 + (p/\rho) - \omega\Psi$. This means that, for a fixed y, $\Phi \to$ const. for all $x > 0$. To evaluate this constant we use the fact that a parabolic system requires an initial condition in order that the mathematical formulation be well-posed. Thus by requiring that

$$u \to U + \omega y, \quad \Psi \to Uy + \tfrac{1}{2}\omega y^2, \quad p \to P \quad \text{as} \quad x \to 0,$$

we have $\Phi \to \frac{1}{2}U^2 + (P/\rho)$ as $x \to 0$; but this result must also be true for *all* $x > 0$, so that the proper asymptotic condition is finally

$$\tfrac{1}{2}u^2 + (p/\rho) - \omega\Psi \to \tfrac{1}{2}U^2 + (P/\rho) \quad \text{as} \quad \bar{y} \to \infty, \tag{2.2}$$

which, along with the no-slip conditions,

$$u = v = 0 \quad \text{at} \quad y = 0, \tag{2.3}$$

completes the 'exact' formulation of the first-approximation system.

As in the classical case, the pressure itself must be specified *a priori* before such a system can be solved. Now on the D_v-scale, (2.1*c*) implies that $\bar{p}$ is in general a function of ξ only; i.e. the pressure within D_v is in general dependent on the nature of the vorticity interaction. We are therefore faced with the necessity of making a plausible assumption regarding the nature of the interaction on the pressure before we can commence with the solution. To this end it is assumed that the undisturbed pressure field is unchanged by the effects of the interaction, i.e. $p = P$ and hence $\partial p/\partial x = 0$ uniformly in D_v to the first approximation. We shall now provide a more rigorous basis for this assumption, other than the physical for the case of strong interaction.

The first-approximation system is accordingly modified by dropping the pressure-gradient term from (2.1*b*) and setting $p = P$ in (2.2) to give

$$u^2 \to U^2 + 2\omega\Psi \quad \text{as} \quad \bar{y} \to \infty. \tag{2.4}$$

On the D_v-scale the solution of this modified system would give Ψ or $(\bar{u}, \bar{v})$ as functions of $(\xi, \bar{y})$. An exact solution for arbitrary ξ is yet to be found, but a series solution for the weak-interaction case ($\xi \ll 1$) has been formulated by Goldstein (1930) and solved numerically by Glauert (1957), and an asymptotic solution for the strong-interaction case ($\xi \gg 1$) has been given in *M*. Also, an approximate solution for arbitrary ξ is obtained in *M*, and this in virtue of the usual approximate momentum-integral method.

For the present purpose it is expedient to re-derive the asymptotic solution for $\xi \gg 1$ in a different form than that given in *M*. First an examination of the approximate solution for $\xi \gg 1$ shows that the solution we seek must be of the form

$$\bar{\Psi} = \xi^{-1} g(\zeta) + o(\xi^{-1}), \tag{2.5}$$

where $\zeta = \xi^{\frac{1}{3}}\bar{y}$. The system governing the determination of $g(\zeta)$ is easily obtained as

$$\left.\begin{aligned} 3g''' + \zeta^2 g'' = 0, \\ g = g' = 0 \quad \text{at} \quad \zeta = 0, \quad 2\zeta g' = 1 + 2g \quad \text{as} \quad \zeta \to \infty, \end{aligned}\right\} \tag{2.6}$$

where primes denote differentiation with respect to ζ. The solution satisfying this system is

$$g(\zeta) = C\int_0^{\zeta}\left[\int_0^{\zeta}\exp\left(-\tfrac{1}{9}\bar{\zeta}^3\right)d\bar{\zeta}\right]d\zeta, \tag{2.7}$$

where $C = [9^{\frac{2}{3}}\Gamma(\frac{5}{3})]^{-1}$, Γ being the gamma function. From this it may be confirmed that the diffusion of surface-generated vorticity, being proportional to $g''(\zeta)$, decays in an exponential manner as D_v^e is approached.

We now observe that the dimensionless quantities ξ and ζ may be rewritten as

$$\xi = \sqrt{(\tilde{x}/Re_L)}, \quad \zeta = \tilde{y}(Re_L/\tilde{x})^{\frac{1}{3}},$$

where $Re_L = UL/\nu$ is the characteristic Reynolds number based on the vortical length L, $\tilde{x} = x/L$, and $\tilde{y} = y/L$. Thus the limit $\xi \to \infty$ is formally equivalent to the limit $Re_L \to 0$ with $\tilde{x}$ fixed ($\tilde{x}$ must be fixed in the domain of validity of the first approximation, i.e. where $Re_x \gg 1$ or $\tilde{x} \gg 1/Re_L$). Also, $\zeta \to 0$ is equivalent to $Re_L \to 0$ with $(\tilde{x}, \tilde{y})$ fixed, so that the existence of a *Stokes flow* is implied in the immediate proximity of the plate and far downstream from the leading edge, which is what one would expect if such a sublayer exists at all. In other words, within D_v, where viscous forces are of the same or greater order of importance as inertia forces, there exists a sublayer adjacent to the plate wherein viscous forces are much greater than inertia forces. We shall call it the *Stokes sublayer* and denote it by D_s.

As for D_v, we can define a similar set of operations for D_s. Let $\tilde{u} = (u - \omega y)/U$, $\tilde{v} = v/U$, and $\tilde{p} = \{(p-P)/U^2\}Re_L$. Thus, the appropriate set of dimensionless dependent and independent quantities for D_s is $(\tilde{u}, \tilde{v}, \tilde{p}; \tilde{x}, \tilde{y}, Re_L)$. We shall refer to a physical system that has been scaled in this manner as being placed on the D_s-scale. We define the Stokes limit as

$$\lim_s\colon\ Re_L \to 0 \quad \text{with } (\tilde{x}, \tilde{y}) \text{ fixed in } D_s.$$

The sequence of operations—which we shall refer to as S_s—is equivalent to S_v with D_v and $\lim_v$ replaced respectively by D_s and $\lim_s$ in S_v.

Hence, by applying S_s to (1.1a, b, c) we obtain the following system of equations governing the flow in D_s:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.8a}$$

$$\frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad \frac{\partial p}{\partial y} = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{2.8b, c}$$

It is seen that this limiting system is equivalent to simply neglecting the inertia terms from (1.1b, c), as was to be expected. Our formulation for the flow within D_s is complete by requiring the solutions of (2.8) to satisfy the same boundary conditions as (1.1d, e).

It can easily be verified that these solutions are of the form:

$$u = \omega y + \frac{U}{\pi}\left(\frac{xy}{x^2+y^2} + \theta\right), \quad v = \frac{U}{\pi}\frac{y^2}{x^2+y^2}, \tag{2.9a, b}$$

$$p - P = \frac{2\mu U}{\pi}\frac{y}{x^2+y^2}, \tag{2.9c}$$

where $\theta = \tan^{-1}(y/x)$ varies in the interval $0 \leqslant \theta \leqslant \pi$ for $y \geqslant 0$ (the ray $\theta = 0$ coincides with the plate). It should be emphasized that (2.9) is only a *first approximation* to the flow within D_s, with the error diminishing as Re_L becomes small—and this holds uniformly for all x such that $\tilde{x} \gg 1/Re_L$.

The lateral extent of D_s may be estimated as follows. By using (2.9), the magnitude of the neglected inertia terms in (1.1*b*) is $O(U\omega y^2/x^2)$ when $y^2 \ll x^2$, while the viscous terms are of $O(\nu Uy/x^3)$ when $y^2 \ll x^2$. Thus the inertia forces are much smaller than the viscous forces when

$$y \ll \nu/(\omega x) \quad \text{or} \quad \tilde{y} \ll 1/(Re_L \tilde{x}).$$

(The same result would be obtained if we had used (1.1*c*), but the inertia and viscous terms in the y-momentum balance would be individually of smaller order of importance compared with their counterparts in the x-momentum balance.) Therefore the Stokes sublayer, which exists for $Re_L \ll 1$, may be defined by the following restrictions on the co-ordinates of a given point within it:

$$D_s\colon\ 1/Re_L \ll \tilde{x} < \infty, \quad 0 \leqslant \tilde{y} \ll 1/(Re_L \tilde{x}).$$

It is seen that the lateral dimension of D_s diminishes when the co-ordinate $\tilde{x}$ becomes large. As a matter of comparison, the thickness of D_s is $\ll x/\xi\, Re_x^{\frac{3}{2}}$, while the thickness of D_v is $\sim x/\xi^{\frac{1}{3}} Re_x^{\frac{1}{2}}$, so that the ratio of the former to the latter is $\ll 1/\xi^{\frac{2}{3}} Re_x$.

Now according to (2.9*c*) the value of the pressure at $y = 0$ is $p = P$, which is independent of the viscosity and the distance from the leading edge. Therefore, since the pressure is constant across D_v and since $p = P$ at $y = 0$, we must have $p = P$ uniformly across D_v, which was to be justified for $\xi \gg 1$. It must be emphasized that this conclusion holds *uniformly* for *all* x such that $\tilde{x} \gg 1/Re_L$ or $Re_x \gg 1$.

3. On the flow below the plate

The previous section was devoted entirely to the flow above the plate, which was treated on the tacit assumption that it was not influenced in any manner by the flow below the plate. We now examine the nature of this bottom flow on the basis of a heuristic approach.

To begin with, let us assume that the scale factors and the limit process valid in D_v also apply in some domain D_b in the vicinity of and below the plate. Let us also assume that the application of S_v to (1.1*a, b, c*) is permissible in this bottom region. Finally, let us assume that $p = P$ is a good approximation within D_b. The governing equations for the flow in D_b are thus (2.1*a, b, c*) with $\partial p/\partial x = 0$. The associated boundary conditions are the no-slip conditions (2.3) and

$$u^2 \rightarrow U^2 + 2\omega\Psi \quad \text{as} \quad \bar{y} \rightarrow -\infty.$$

For the present purpose an approximate solution of this system is sufficient. It will be seen that, by a suitable interpretation, the approximate solution (obtained from the momentum-integral method) given in M applies in D_b, the essential result of which is

$$1 + (\Lambda - 1)(2\Lambda + 1)^{\frac{1}{2}} = \tfrac{9}{2}\xi^2,\dagger \tag{3.1}$$

† Due to a typographical lapse the term of unity was omitted in M. Equation (4.7) of M should read like (3.1) above.

where $\Lambda = \omega\Psi_\delta/U^2$, Ψ_δ being the dimensional stream function at a location where the diffusion of surface-generated vorticity has practically decayed to zero.

By transposing the unity term to the right-hand side of (3.1) and squaring the result, we obtain

$$2\Lambda^3 - 3\Lambda^2 + 1 = (\tfrac{9}{2}\xi^2 - 1)^2, \tag{3.2}$$

which, for a given ξ, is a cubic equation. However, not all three roots of (3.2) have relevance to this problem, i.e. are solutions of (3.1). For example, the roots for $\xi = 0$ are $(\frac{3}{2}, 0, 0)$, of which only $\Lambda = \frac{3}{2}$ does not satisfy (3.1); for $\xi = \frac{1}{3}$, only $\Lambda = (0{\cdot}67, -0{\cdot}44)$ have relevance, where $\Lambda = 0{\cdot}67$ is the result for $y > 0$ and $\Lambda = -0{\cdot}44$ is here interpreted as being valid for $y < 0$; for $\xi = \sqrt{2}/3$, $\Lambda = (1, -\frac{1}{2})$ have relevance, where $\Lambda = 1$ applies in $y > 0$ and $\Lambda = -\frac{1}{2}$ applies in $y < 0$; for $\xi = \frac{2}{3}$, only $\Lambda = \frac{3}{2}$ has relevance and it applies only in $y > 0$. It is thus evident that, beyond a certain $\xi = \xi_c$, no solutions of (3.1) exist in $y < 0$.

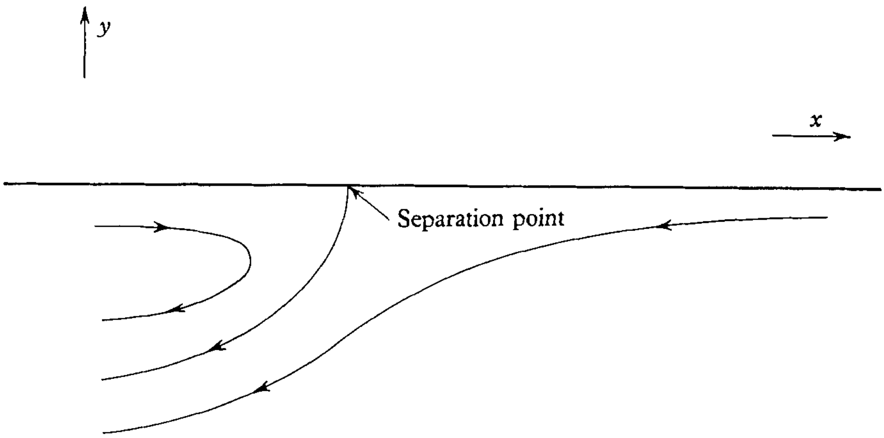

FIGURE 2. Possible streamline pattern below the plate.

This critical value ξ_c may be easily determined as follows. In M the approximate expressions for the skin friction τ_w and the velocity components (u_δ, v_δ) at the distance $y = \delta$ corresponding to Ψ_δ are given by

$$\left.\begin{aligned} \tau_w\left(\frac{x}{\rho u U^3}\right)^{\frac{1}{2}} &= \xi\left(1+\frac{1}{2\Lambda}\right), \quad u_\delta = U(1+2\Lambda)^{\frac{1}{2}}, \\ v_\delta\left(\frac{x}{U\nu}\right)^{\frac{1}{2}} &= \frac{3}{2}\frac{\xi}{\Lambda(1+2\Lambda)^{\frac{1}{2}}}, \quad \delta\left(\frac{U}{\nu x}\right)^{\frac{1}{2}} = \frac{2\Lambda}{\xi(1+2\Lambda)^{\frac{1}{2}}}. \end{aligned}\right\} \tag{3.3}$$

It is thus seen that as $\Lambda \to -\frac{1}{2}$, i.e. as $\xi \to \sqrt{2}/3$, $\tau_w \to 0$, $u_\delta \to 0$, $v_\delta \to \infty$, and $\delta \to \infty$. Thus it is suggested here that these limiting values are symptomatic of *flow separation* and that $\xi_c = \sqrt{2}/3$ is a point of separation of the flow from the lower surface of the plate.

Physically, this means that the kinetic energy of the viscous fluid in its downstream motion near the plate is insufficient to overcome the work done on it by the 'dragging effect', acting in the opposite (upstream) direction, of the fluid far from the plate in $y < 0$. An impasse is reached at approximately $\xi_c = \sqrt{2}/3$, where the forward motion of a fluid particle near the plate comes to a dead halt and subsequently has its direction reversed (figure 2).

The existence of flow separation below the plate means that the initial assumptions, adopted on a heuristic basis, cannot be justified. In other words, the mathematical system adopted for the representation of the flow below the plate is not accurate in the least. Certainly surface-generated vorticity is no longer confined to a thin layer adjacent to the plate, rather it leaves the surface along streamlines that penetrate sharply into the outer flow. Moreover, there is a tendency for surface-generated vorticity to be transported upstream along streamlines that have been reversed in direction, so that there is a strong possibility of the generation of an upstream wake. (It is surmised that, since the surface-generated vorticity diffuses as it is convected upstream, its net effect on the flow far upstream is negligible.) Finally, since the undisturbed streamline pattern has been severely disrupted, the assumption that $p = P$ in D_b is untenable. Thus a substantial advance in the theory must be made before the exact nature of the separated flow phenomenon and its subsequent effects on the flow can be fully understood.

Since we are considering an infinitely long plate, the separation phenomenon is, relatively speaking, a *local* one—occurring close to the leading edge. At large distances downstream such a local phenomenon must 'die out', so that the flow eventually settles down to a more regular behaviour. This is ascertained by the deduction that a Stokes flow can be found which satisfies (2.8) and (1.1 d, e) in $y \leqslant 0$; it is of the form

$$\left.\begin{aligned} u = \omega y - \frac{U}{\pi}\left(\frac{xy}{x^2+y^2}+\theta\right), \quad v = -\frac{U}{\pi}\frac{y^2}{x^2+y^2}, \\ p-P = -\frac{2\mu U}{\pi}\frac{y}{x^2+y^2}, \end{aligned}\right\} \qquad (3.4)$$

where θ varies in the interval $-\pi \leqslant \theta \leqslant 0$. Thus (3.4) describes the flow very close to the plate, where inertia forces are much smaller than viscous forces. More precisely, such a flow exists when $Re_L \ll 1$ and is uniformly valid in the domain $x \gg x_c$ and $|y| \ll \nu/\omega x$, where x_c is the point of separation of the flow from the lower surface of the plate. Note that, since $v \leqslant 0$, the undisturbed streamlines have been shifted outwards from the plate in the negative y-direction due to the action of viscosity.

4. Discussion

The analysis given in § 2 justifies on a rigorous basis the strong-interaction solution given in M for the flow in D_v—i.e. it is in fact a limiting solution of system (1.1). In this concluding section we shall examine a different description of the flow in D_v in the light of this analysis. We shall first find it expedient to exhibit in a general manner the physical situation.

Consider an elemental volume ΔV fixed at D_v^e. The velocity components there are (u_δ, v_δ). Owing to the retarding action of viscosity, the positive quantity $\Delta M_y = \rho v_\delta \omega \Delta V$ represents that part of the total net tangential momentum of the fluid removed from ΔV in unit time with the mass flux ρv_δ in the positive y-direction, the total net amount removed being $\rho\{\partial(u_\delta v_\delta)/\partial y\}\Delta V$. (Here we neglect the small net contribution of tangential momentum removed from ΔV

by a molecular process which transfers tangential momentum flux from points of high velocity to points of low velocity.) In order to conserve momentum in ΔV in the tangential direction, ΔM_y must be balanced by a net decrease of amount $(\Delta M_x + \Delta p)$ in ΔV per unit time, where $\Delta M_x = -\rho u_\delta(\partial u_\delta/\partial x)\Delta V$ represents half of the total net decrease in tangential momentum that is transported with the mass flux ρu_δ, and $\Delta p = -(\partial p/\partial x)\Delta V$ represents the net decrease in the pressure force acting on ΔV in the tangential direction.

The physical point of view adopted by Li (1956), Murray (1961) and Van Dyke (1962) for the weak interaction case and by Ting (1960) for the intermediate and strong interaction cases is that ΔM_y is balanced entirely by Δp, where the contribution ΔM_x is negligible (i.e. there is no acceleration of the fluid in the tangential direction) and Δp is due to a favourable pressure gradient regarded as being induced by the vorticity interaction.† However, when such a physical argument is tested in the situation wherein the vorticity interaction is intense—i.e. against the rigorous results of § 2, it fails to meet the test since that analysis does not produce such a pressure gradient. Thus the corresponding mathematical solution cannot be a limiting solution of system (1.1) as we have posed it.

The actual physical situation in the strong interaction case must be such that ΔM_y is balanced entirely by ΔM_x in ΔV, i.e. all the changes are associated with the mass motion. Thus the presence of external shear at D_v^e provides a 'medium' for the continual removal of tangential momentum away from D_v in the vertical direction, the consequence being that the removed momentum cannot contribute to the skin friction drag. Owing to the monotonic behaviour of ρv_δ with respect to x, it is reasonable to infer that this represents the correct physics of the situation for all interaction intensities.

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† This point of view is supported by Toomre & Rott (1964) who examined in part the unbounded flow external to D_v on the basis of a symmetrical inviscid flow model. This model has however several drawbacks: (i) it is vulnerable to questions of uniqueness since the oncoming vorticity distribution is discontinuous; (ii) it does not simulate all the essential phenomena—e.g. an inviscid fluid cannot transfer momentum irreversibly (in the thermodynamic sense) when vorticity is present, and (iii) it cannot accommodate the possibility of tangential disturbances occurring at the boundaries and their resultant transmission into the flow (see § 1).